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NIEIZOMORFICZNE DRZEWA W KLIKACH K_7 ORAZ K_8

Streszczenie: W pracy zestawiono rodziny nieizomorficznych drzew w klikach K_8 oraz K_7. Zestawiono ich ciągi stopni. Wykazano brak izomorfizmu dla drzew mających ten sam ciąg stopni analizując ciągi stopni sąsiadów.

Słowa kluczowe: ciąg stopni wierzchołków, enumeracja

NON-ISOMORPHIC TREES IN CLIQUES K_7 AND K_8

Summary: In the present paper, the families of non-isomorphic trees in cliques K_8 and K_7 are presented. The vertices' degrees sequences are listed. Lack of isomorphisms was confirmed for chosen trees having the same degree sequences, analysing the degrees of neighbour vertices.

Keywords: sequence of vertices' degrees, enumeration

1. Introduction

Graph theory is a branch of discrete mathematics [1,4] which has versatile applications in many fields of science, technology and other human activities. These applications are especially important nowadays when the world is covered via different networks. A network is a weighted graph in which the vertices, called as nodes, have fixed positions e.g. geographical co-ordinates. However, graph itself has not fixed positions of its vertices and it could be graphically presented in different

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ways. Some applications of isomorphism detection are described in paper [2] where non-isomorphic layouts of modular robots are considered.

Graph theory had been originated via the paper of Leonhard Euler published in 1736 - about the Königsberg bridges [1]. Today this town belongs to Russia and it has a new name, namely: Kaliningrad.

Graph G(V,E) is defined as a pair of sets i.e.: V - set of vertices and E - set of edges. Set V should be non-empty. So, at least one vertex exists. Set E could be empty therefore in this case the graph consists just of separate vertices.

In a graph, we can consider different subgraphs e.g.: paths, cuts or cycles. Subgraph G'(V', E') is a part of a graph G(V, E) which means that V' is a subset of V and E' is a subset of E. We usually consider the following notations: $|V| = n$ is a number of vertices and $|E| = m$ is a number of edges.

Cycle is a sequence by turns of vertices and edges (connecting neighboring vertices) in which the end vertex is equal to the start vertex.

Path is a similar sequence as mentioned above but the start and end vertices are different.

Cut is a set of edges which removal from a graph causes that the graph is split into separate parts – it is divided at least into two separate components – so called: connectivity components.

In the present paper, we will consider the problem of enumeration of some families of trees. During the previous conference in 2015, the problem of enumeration of a family of graphs having the same sequence of vertices' degrees was considered [5]. Professional considerations on enumeration of trees can be found in paper [3] where also the same tree families are listed, but in a different order than in the present work due to considerations of more sophisticated tree indicators than a sequences of degrees.

2. Tree T – subgraph of graph G

Tree $T(V_T,E_T)$ is a subgraph of a graph G(V,E). In case when $V_T = V$, a tree is called a spanning tree. It means that the spanning tree has the same set of vertices as a source graph but usually the word spanning is omitted and the scientists know what is consideration about. The definition of tree is as follows: it is a graph being connected and acyclic. Therefore, in consequence, spanning tree has $n – 1$ edges [1].

In common sense, it can be said that a graph is connected when it is built of one entity. In other words, there is exactly one path between every two arbitrary vertices. Connectivity is an immanent feature of a tree so every tree edge is a bridge [1]. Bridge is a cut consisting of one edge only. So after removal of an arbitrary edge from a tree – the tree losses the connectivity feature and it is divided into two connectivity components. In case if the end vertex of a removed edge has degree one then such edge is called a leaf.

Isomorphism of two graphs can be confirmed when the existence of bijection between vertices is equivalent to an existence of a relevant bijection for its edges. These bijections are mutually coupled. In common sense isomorphic graphs have the same shape or geometrical layout taking into account that positions of graph vertices are arbitrary and the same graph can be drawn in infinite manners. However, its adjacency matrix remains the same. Checking of isomorphism of trees is a solved

problem. Three (3) known algorithms were implemented in a software prepared within the framework of the master thesis at our University and they are described in paper [6], together with the results of utilization of the prepared software.
Number of all possible trees in a clique is expressed by Cayley formula [1,4]:

$$\text{Number\_of\_trees\_of\_K}_n = n^{n-2} \quad (1)$$

For example, for consecutive n – we have the following numbers of trees: 2→1 because $2^0 = 1$, 3→3 because $3^1 = 3$, 4→16 because $4^2 = 16$ and 5→125 because $5^3 = 125$ etc.
Whereas the numbers of non-isomorphic trees are given in Table 1.

*Table 1. Numbers of non-isomorphic trees in cliques K_n **[4]***

K_n; n	2	3	4	5	6	7	8	9	10	11
No non-isomorph. trees	1	1	2	3	6	11	23	47	106	235

For example for clique K_4, we have two non-isomorphic trees from all 16 possible ones /based upon formula (1) because $4^2 = 16$/ i.e.:
- a star-like tree and
- a path P_3 having the length 3 because it has 3 edges.

3. Families of trees for chosen cliques

Like it is mentioned in the paper title, underneath the families of non-isomorphic trees for K_7 and K_8 are shown graphically.

The graphs were obtained via a method of trials and errors. Obviously, it is not a general approach but it is worth to discuss within a framework of student project. In Fig. 1, the family of no-isomorphic trees in the clique K_7. Like previously, tree T_1 is a star-like graph – just a star and tree T_2 is a path P_6 because it contains 6 edges.
Underneath, the tables 2 and 3 are shown where the rearranged degree sequences are listed. Every sequence is presented in the following manner: from the highest degrees down to the lowest ones. So, every sequence has 7 positions and 8 positions, respectively.

Table 2. Degree sequences for the trees presented in Figure 1 (clique K_7)

Tree	Sequence of vertices' degrees	Tree	Sequence of vertices' degrees
T_1	(6,1,1,1,1,1,1)	T_7	(4,3,1,1,1,1,1)
T_2	(2,2,2,2,2,1,1)	T_8	(4,2,2,1,1,1,1)
T_3	**(3,2,2,2,1,1,1)**	T_9	**(3,2,2,2,1,1,1)**
T_4	**(3,2,2,2,1,1,1)**	T_{10}	(3,3,2,1,1,1,1)
T_5	(5,2,1,1,1,1,1)	T_{11}	(4,2,2,1,1,1,1)
T_6	(3,3,2,1,1,1,1)		

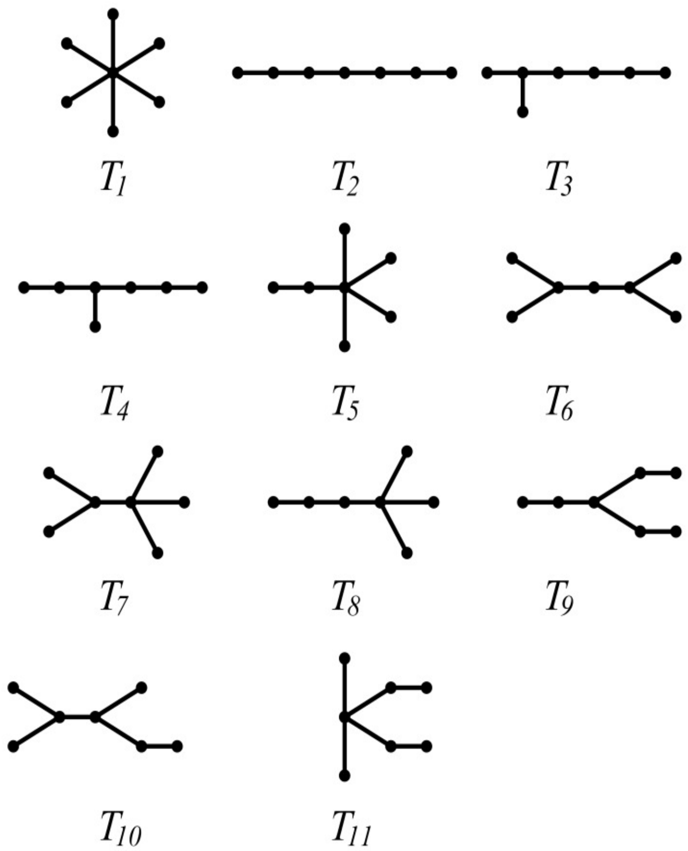

Figure 1. Non-isomorphic trees in clique K_7

Table 3. Degree sequences for the trees presented in Figure 2(clique K_8)

Tree	Sequence of vertices' degrees	Tree	Sequence of vertices' degrees
D_1	(7,1,1,1, 1,1,1,1)	D_{13}	(4,3,2,1, 1,1,1,1)
D_2	(2,2,2,2, 2,2,1,1)	D_{14}	(5,2,1,1, 1,1,1,1)
D_3	(6,2,1,1, 1,1,1,1)	D_{15}	(4,3,2,1, 1,1,1,1)
D_4	(3,2,2,2, 2,1,1,1)	D_{16}	(3,3,3,1, 1,1,1,1)
D_5	(3,2,2,2, 2,1,1,1)	D_{17}	(4,2,2,2, 1,1,1,1)
D_6	(3,2,2,2, 2,1,1,1)	D_{18}	(3,2,2,2, 2,1,1,1)
D_7	(4,2,2,2, 1,1,1,1)	D_{19}	(3,3,3,1, 1,1,1,1)
D_8	**(5,2,2,1, 1,1,1,1)**	**D_{20}**	**(5,2,2,1, 1,1,1,1)**
D_9	(3,3,2,2, 1,1,1,1)	D_{21}	(3,3,2,2, 1,1,1,1)
D_{10}	(3,3,2,2, 1,1,1,1)	D_{22}	(4,4,1,1, 1,1,1,1)
D_{11}	(3,3,2,2, 1,1,1,1)	D_{23}	(3,3,2,2, 1,1,1,1)
D_{12}	(4,2,2,2, 1,1,1,1)		

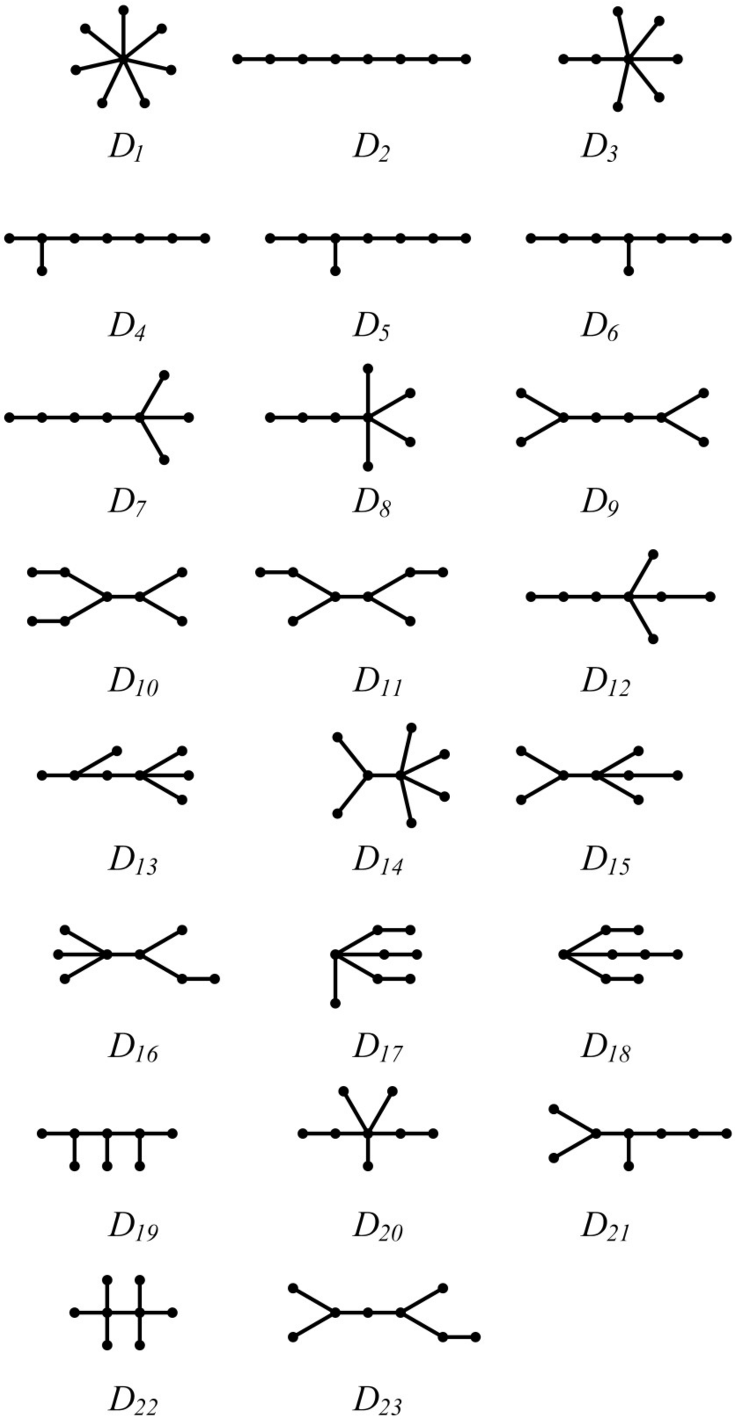

Figure 2. Non-isomorphic trees in clique K_8

In case of the Table 3, every sequence has obviously 8 positions i.e. 8 elements in every sequence.

According to the general graph-theoretical properties, the sum of elements of every sequence is equal to: $2 \times m = 2 \times (n - 1) = 2 \times n - 2$; the last equality is proper for every tree. Therefore, for $n = 7$ we have $2 \times n - 2 = 2 \times 7 - 2 = 12$; e.g. for the tree T_1 we have $6 + 1 + 1 + 1 + 1 + 1 + 1 = 12$ and for tree D_1 (see Table 3) the calculations are as follows: $2 \times m = 2 \times 7 = 14 = 7 + 1 + 1 + 1 + 1 + 1 + 1 + 1$.
Additional feature of these sequences is that the number of odd numbers is even.

One can see that some degree sequences are the same despite the fact that only non-isomorphic trees have been drawn and listed. It just confirms that these codes are not unique. It is also known that - so called - Prűfer's code is fully representative for trees [7].

The problem of effective coding of a tree depends on the considered problem as well as on the applied algorithm. In case of the problem of minimal spanning tree with constraints which was solved utilizing evolutionary algorithm – a set of edges was proposed as an effective tree code [8,9].

4. Lack of isomorphism for some pairs of trees having the same sequence of vertices' degrees

Like it was mentioned above some trees have the same degree sequences. However, it can be shown that they are non-isomorphic. In case of paper [5] such proving was performed via listing of cycles families of particular graphs. But, this approach is useless in our problem because a tree does not poses any cycle - based upon its definition. Other way of showing lack of isomorphism is checking identity of the lists of degrees of neighbors of consecutive vertices. Lack of identity means that these trees (or graphs, in general) are non-isomorphic.

Let's consider two exemplary trees D_8 and D_{20} which degree sequences (5,2,2,1, 1,1,1,1) are highlighted in bold characters in Table 3. Analyzing the pictures of these graphs shown in Figure 2, we can write the sequences of degrees of their neighbor vertices – see Table 4 – underneath.

Table 4. Chosen degree sequences for neighbor vertices for trees D_8 and D_{20} presented in Figure 2

Vertex degree	Sequence of degrees of neighbors in **D_8**	Sequence of degrees of neighbors in **D_{20}**
5	2,1,1,1,1	2,2,1,1,1
2	5,2	5,1
2	2,1	5,1

Based upon the data gathered in Table 4, we can see that despite the fact of <u>identity</u> of the sequence of vertices' degrees for both considered trees (D_8 and D_{20}), the adequate sequences of degrees of neighbor vertices are different. Is even enough to analyze the vertex of degree 5 which is only one in every tree.

Moreover, the sequences of degrees of neighbor vertices for two vertices of degree 2 (two in every case) are also inconsistent. In this way we prove that the trees D_8 and D_{20} are really different in shape as well as non-isomorphic which – on other hand – can be easily seen upon the eye examination (visual inspection).

5. Conclusions

In the present paper the problem of non-isomorphic trees in cliques has been discussed.

The considerations were performed within the project of ERASMUS students.

The families of these trees are listed for cliques K_7 and K_8. The obtained results are consistent with these presented in paper [3]. It confirms correctness of the outcomes, however the order and the layouts are different – what also confirms own attitude to the task

It was shown that some trees having the same sequence of vertices' degrees are not isomorphic via checking the degrees of their neighbors. The method used in [5] consisting in comparison of cycles families for compared graphs is here pointless because every tree is acyclic.

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